St. Venant-Kirchhoff Membrane Energy Eigensystem

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The St. Venant-Kirchhoff membrane energy density Ψ is defined as

$$\Psi(\mathbf{F}) = \mu\mathbf{E} : \mathbf{E} + \frac{\lambda}{2}\operatorname{tr}^2(\mathbf{E}), \tag{1}$$

where $\mathbf{F} \in \mathbb{R}^{3\times 2}$ is the in-plane deformation gradient, $\mathbf{E} = \frac{1}{2}(\mathbf{F}^\top\mathbf{F} - \mathbf{I}) \in \mathbb{R}^{2\times 2}$ is the in-plane Green strain tensor and $\mathbf{I}$ is the 2×2 identity matrix.

Following the methods in [Kim 2020; Panetta 2020; Smith et al. 2019], we first convert this $\mathbf{F}$-based representation into an $\mathbf{S}$-centric-invariant-based representation:

$$\begin{aligned}\Psi &= \mu(\frac{1}{4}(\sigma_1^2 - 1)^2 + \frac{1}{4}(\sigma_2^2 - 1)^2) + \frac{\lambda}{2}(\frac{1}{2}(\sigma_1^2 - 1) + \frac{1}{2}(\sigma_2^2 - 1))^2 \\ &= \frac{\mu}{4}(\sigma_1^4 + 1 - 2\sigma_1^2 + \sigma_2^4 + 1 - 2\sigma_2^2) + \frac{\lambda}{8}(\sigma_1^2 + \sigma_2^2 - 2)^2 \\ &= \frac{\mu}{4}(I_2^2 - 2I_3^2 + 2 - 2I_2) + \frac{\lambda}{8}(I_2 - 2)^2 \\ &= \frac{\lambda + 2\mu}{8}I_2^2 - \frac{\lambda + \mu}{2}I_2 - \frac{\mu}{2}I_3^2 + \frac{\lambda + \mu}{2},\end{aligned} \tag{2}$$

where σ_1 and σ_2 are two singular values of $\mathbf{F}$ obtained from the singular value decomposition $\mathbf{F} = \mathbf{U}\begin{bmatrix}\sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top$, $I_1 := \sigma_1 + \sigma_2$, $I_2 := \sigma_1^2 + \sigma_2^2$ and $I_3 := \sigma_1\sigma_2$ are three singular-value-centric invariants. Using the chain rule, the second derivative of Ψ with respect to $\mathbf{F}$ can be written as below:

$$\frac{\partial^2\Psi}{\partial\mathbf{F}^2} = \frac{\lambda + 2\mu}{4}(\frac{\partial I_2}{\partial\mathbf{F}} \otimes \frac{\partial I_2}{\partial\mathbf{F}} + I_2\frac{\partial^2 I_2}{\partial\mathbf{F}^2}) - \frac{\lambda + \mu}{2}\frac{\partial^2 I_2}{\partial\mathbf{F}^2} - \mu(\frac{\partial I_3}{\partial\mathbf{F}} \otimes \frac{\partial I_3}{\partial\mathbf{F}} + I_3\frac{\partial^2 I_3}{\partial\mathbf{F}^2}) \tag{3}$$

We use the definition $a := \frac{\lambda+2\mu}{4}I_2 - \frac{\lambda+\mu}{2}$, $b := \frac{\lambda+2\mu}{4}$, $c := -\mu$ and $d := -\mu I_3$ for conciseness and the second derivative can be further expressed analytically based on the conclusions in [Panetta 2020] as below:

$$\frac{\partial^2\Psi}{\partial\mathbf{F}^2} = a\underbrace{\frac{\partial^2 I_2}{\partial\mathbf{F}^2}}_{2\mathbf{I}_{6\times 6}} + b\underbrace{\frac{\partial I_2}{\partial\mathbf{F}}}_{2\mathbf{F}} \otimes \frac{\partial I_2}{\partial\mathbf{F}} + c\underbrace{\frac{\partial I_3}{\partial\mathbf{F}}}_{\mathbf{U}\begin{bmatrix}\sigma_2 & 0 \\ 0 & \sigma_1 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top} \otimes \frac{\partial I_3}{\partial\mathbf{F}} + d\frac{\partial^2 I_3}{\partial\mathbf{F}^2}. \tag{4}$$

Clearly, any orthogonal basis can be chosen as a set of eigenvectors for $\frac{\partial^2 I_2}{\partial\mathbf{F}^2}$. Because both $\frac{\partial I_2}{\partial\mathbf{F}}$ and $\frac{\partial I_3}{\partial\mathbf{F}}$ are orthogonal to any vector with such a pattern $w\mathbf{U}\begin{bmatrix}0 & x \\ y & 0 \\ z & v\end{bmatrix}\mathbf{V}^\top$, we can immediately get four eigenvalues and corresponding eigenvectors for $\frac{\partial^2\Psi}{\partial\mathbf{F}^2}$ with the help of the analytic eigensystem for $\frac{\partial^2 I_3}{\partial\mathbf{F}^2}$ in [Panetta 2020]:

$$\begin{aligned}&\underbrace{\mathbf{e}_1 = \frac{1}{\sqrt{2}}\mathbf{U}\begin{bmatrix}0 & -1 \\ 1 & 0 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top,}_{\lambda_1 = 2a+d} \quad \underbrace{\mathbf{e}_2 = \frac{1}{\sqrt{2}}\mathbf{U}\begin{bmatrix}0 & 1 \\ 1 & 0 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top,}_{\lambda_2 = 2a-d} \\ &\underbrace{\mathbf{e}_3 = \mathbf{U}\begin{bmatrix}0 & 0 \\ 0 & 0 \\ 1 & 0\end{bmatrix}\mathbf{V}^\top,}_{\lambda_3 = 2a+\frac{\sigma_2}{\sigma_1}d} \quad \underbrace{\mathbf{e}_4 = \mathbf{U}\begin{bmatrix}0 & 0 \\ 0 & 0 \\ 0 & 1\end{bmatrix}\mathbf{V}^\top.}_{\lambda_4 = 2a+\frac{\sigma_1}{\sigma_2}d}\end{aligned} \tag{5}$$

The left two eigenpairs can be found by firstly projecting $\frac{\partial^2\Psi}{\partial\mathbf{F}^2}$ into the space spanned by $\frac{\partial I_2}{\partial\mathbf{F}}$ and $\frac{\partial I_3}{\partial\mathbf{F}}$ and then finding the eigensystem of the reduced Hessian. Following the method in [Panetta 2020], we compute this reduced Hessian by selecting two simple bases spanning this space, $\mathbf{D}_1 = \mathbf{U}\begin{bmatrix}1 & 0 \\ 0 & 0 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top$ and $\mathbf{D}_2 = \mathbf{U}\begin{bmatrix}0 & 0 \\ 0 & 1 \\ 0 & 0\end{bmatrix}\mathbf{V}^\top$:

$$\begin{aligned}&\begin{bmatrix}\mathbf{D}_1 : \frac{\partial^2\Psi}{\partial\mathbf{F}^2} : \mathbf{D}_1 & \mathbf{D}_1 : \frac{\partial^2\Psi}{\partial\mathbf{F}^2} : \mathbf{D}_2 \\ \mathbf{D}_2 : \frac{\partial^2\Psi}{\partial\mathbf{F}^2} : \mathbf{D}_1 & \mathbf{D}_2 : \frac{\partial^2\Psi}{\partial\mathbf{F}^2} : \mathbf{D}_2\end{bmatrix} \\ &= 2a\begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix} + 4b\begin{bmatrix}\sigma_1^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_2^2\end{bmatrix} + c\begin{bmatrix}\sigma_2^2 & \sigma_1\sigma_2 \\ \sigma_1\sigma_2 & \sigma_1^2\end{bmatrix} + d\begin{bmatrix}0 & 1 \\ 1 & 0\end{bmatrix} \\ &:= \begin{bmatrix}\alpha & \gamma \\ \gamma & \beta\end{bmatrix}.\end{aligned} \tag{6}$$

With the further definition $Z := \sqrt{(\alpha - \beta)^2 + 4\gamma^2}$, the final two eigenpairs can be expressed analytically as below:

$$\begin{aligned}\lambda_5 &= \frac{\alpha + \beta + Z}{2}, \mathbf{e}_5 = \mathbf{U}\begin{bmatrix}\frac{2\gamma}{\sqrt{4\gamma^2+(\beta-\alpha+Z)^2}} & 0 \\ 0 & \frac{\beta-\alpha+Z}{\sqrt{4\gamma^2+(\beta-\alpha+Z)^2}} \\ 0 & 0\end{bmatrix}\mathbf{V}^\top, \\ \lambda_6 &= \frac{\alpha + \beta - Z}{2}, \mathbf{e}_6 = \mathbf{U}\begin{bmatrix}\frac{2\gamma}{\sqrt{4\gamma^2+(\beta-\alpha-Z)^2}} & 0 \\ 0 & \frac{\beta-\alpha-Z}{\sqrt{4\gamma^2+(\beta-\alpha-Z)^2}} \\ 0 & 0\end{bmatrix}\mathbf{V}^\top.\end{aligned} \tag{7}$$

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